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SIMPLICIAL VOLUME OF $\mathbb{Q}$-RANK ONE LOCALLY SYMMETRIC MANIFOLDS COVERED BY THE PRODUCT OF $\mathbb{R}$-RANK ONE SYMMETRIC SPACES

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ABSTRACT. In this paper, we show that the simplicial volume of $\mathbb{Q}$-rank one locally symmetric spaces covered by the product of $\mathbb{R}$-rank one symmetric spaces is strictly positive.

1. Introduction

The simplicial volume of a connected, oriented manifold M was introduced by Gromov [8]. This is a topological invariant in $\mathbb{R}_{\geq 0}$ and measures how efficiently the fundamental class of M can be represented by simplices. Gromov conjectured that non-positively curved closed manifolds with negative Ricci curvature have positive simplicial volume.

First, the positivity of the simplicial volume was verified for closed negatively curved manifolds by Thurston [17], Gromov [8] and Inoue-Yano [9]. It was verified for closed locally symmetric spaces covered by $SL_3(\mathbb{R})/SO_3(\mathbb{R})$ by Savage [16] and Bucher-Karlsson [2]. Then, Lafont and Schmidt [10] showed that the simplicial volume of all closed locally symmetric spaces of non-compact type is positive, which gave a positive answer to the conjecture raised by Gromov.

For open manifolds, the simplicial volume is somewhat mysterious. Thurston [17] verified that the simplicial volume of complete Riemannian manifolds with pinched negative sectional curvature and finite volume is strictly positive. In contrast, Gromov [8], Löh and Sauer [15] proved that the simplicial volume of open manifolds, which are the Cartesian product of three open manifolds and locally symmetric spaces of $\mathbb{Q}$-rank at least 3, vanishes. Löh and Sauer [14] showed that Hilbert modular varieties have positive simplicial volume, which was the first class of examples of open locally symmetric spaces of $\mathbb{R}$-rank at least 2 for which the positivity of simplicial or minimal volume is known. Hilbert modular varieties are special cases of $\mathbb{Q}$-rank one locally symmetric spaces covered by the product of hyperbolic planes. The aim of this paper is to show the positivity of the simplicial volume of $\mathbb{Q}$-rank one locally symmetric spaces covered by the product of $\mathbb{R}$-rank one symmetric spaces.

Theorem 1.1. *Let M be a $\mathbb{Q}$-rank one locally symmetric space covered by the product of $\mathbb{R}$-rank one symmetric spaces. Then, the simplicial volume of M is positive.*

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For general $\mathbb{Q}$-rank one locally symmetric spaces, see the forthcoming paper [3]. Gromov [8] proved a lower bound for the minimal volume of n-dimensional Riemannian manifolds in terms of the simplicial volume:

$$||M|| \leq (n-1)^n n! \cdot \mathrm{minvol}(M).$$

The theorem implies the positivity of minimal volume of $\mathbb{Q}$-rank one locally symmetric spaces covered by the product of $\mathbb{R}$-rank one symmetric spaces as follows. See Connell and Farb [4] for different approach using Lipschitz class of locally symmetric metrics.

Corollary 1.2. *The minimal volume of $\mathbb{Q}$-rank one finite volume locally symmetric spaces covered by the product of $\mathbb{R}$-rank one symmetric spaces is positive.*

So far, the degree theorem for the open locally symmetric spaces of non-compact type with finite volume holds with the Lipschitz condition on the map. From Theorem 1.1, one can obtain the degree theorem without any Lipschitz condition on the map.

Theorem 1.3. *Let N be a Riemannian n-dimensional manifold of finite volume with Ricci curvature bounded below by $-(n-1)$ and M be a $\mathbb{Q}$-rank one locally symmetric space covered by the product of $\mathbb{R}$-rank one symmetric spaces. For any proper map $f : N \to M$ we have*

$$deg(f) \leq C_n \frac{vol(N)}{vol(M)},$$

where C_n depends only on n.

The degree theorem for general locally symmetric space of finite volume is proved by Connell and Farb [4, 5] with Lipschitz condition on f. The essential part of our approach is to show that the geodesic straightening map is well-defined on the locally finite chain complex of $\mathbb{Q}$-rank one locally symmetric spaces. In fact, Thurston [17] introduced the geodesic straightening map on the singular chain complex of non-positively curved manifolds, which is homotopic to the identity.

Unfortunately, the geodesic straightening map is generally not defined on the locally finite chain complex of non-positively curved manifolds because the geodesic straightening of a locally finite chain is not necessarily locally finite. However, the situation in the $\mathbb{Q}$-rank one locally symmetric spaces is different. By using Leuzinger's explicit geometric description of $\mathbb{Q}$-rank one locally symmetric spaces [11], one can see that the geodesic straightening of a locally finite chain is locally finite. The presence of the geodesic straightening map on a locally finite chain complex and the uniform upper bound of the volume of geodesic simplices in $\mathbb{Q}$-rank one locally symmetric spaces covered by the product of $\mathbb{R}$-rank one symmetric spaces give rise to the positivity of the simplicial volume.

2. Preliminaries

In this section, we first collect some definitions and results about the simplicial volume. We begin with the definition of the simplicial volume.

2.1. **Simplicial volume.** Let M be an n-dimensional, connected, oriented manifold. Denote by $C_*(M)$ the singular chain complex of M with real coefficients. Consider on $C_*(M)$ the ℓ^1-norm with respect to the canonical basis of singular simplices, that is, $\|c\|_1 = \sum_{i=1}^r |a_i|$ for $c = \sum_{i=1}^r a_i\sigma_i$ in $C_*(M)$. This norm induces a semi-norm on the real coefficient homology $H_*(M)$ of M as follows:

$$\|z\| = \inf_c \|c\|_1,$$

where c runs over all singular cycles representing $z \in H_*(M)$.

The simplicial volume $\|M\|$ of a closed manifold M is defined as the semi-norm of the fundamental class $[M] \in H_n(M)$.

For closed Riemannian manifolds, Gromov proved the remarkable proportionality principle relating the simplicial volume and the volume of the Riemannian manifolds [8].

Theorem 2.1 (Gromov)**.** *Let M and N be two closed Riemannian manifolds with isometric universal covers. Then,*

$$\frac{\|M\|}{vol(M)} = \frac{\|N\|}{vol(N)}.$$

If M is open, one cannot choose the fundamental class of M in the singular homology of M because the top dimensional homology of M is trivial. Hence, the simplicial volume of open manifolds is defined in terms of the locally finite chain complex of the manifolds as follows.

Let M be a topological space and let $S_k(M)$ be the set of all continuous maps from the standard k-simplex Δ^k to M. A subset A of $S_k(M)$ is called *locally finite* if any compact subset of M intersects the image of only a finite number of elements of A. Let us denote by $S_k^{\mathrm{lf}}(M)$ the set of all locally finite subsets of $S_k(M)$.

Definition 2.2. Let M be a topological space and let $k \in \mathbb{N}$. The locally finite chain complex of M is the chain complex $C_*^{\mathrm{lf}}(M)$ consisting of the real vector spaces

$$C_k^{\mathrm{lf}}(X) = \left\{ \sum_{\sigma\in A} a_\sigma \cdot \sigma \;\middle|\; A \in S_k^{\mathrm{lf}}(X) \text{ and } (a_\sigma)_{\sigma\in A} \subset \mathbb{R} \right\}$$

equipped with the boundary operator given by the alternating sums of the $(k-1)$-faces. The locally finite homology $H_*^{\mathrm{lf}}(M)$ of M is the homology of the locally finite chain complex $C_*^{\mathrm{lf}}(M)$.

The ℓ^1-norm $\|\cdot\|_1$ on the locally finite chain complex of M is defined with respect to the canonical basis of singular simplices, and it also gives rise to a semi-norm on the locally finite homology of M. Any oriented, connected manifold M possesses a fundamental class, which is a distinguished generator of the top dimensional locally finite homology $H_n^{\mathrm{lf}}(M;\mathbb{Z}) \cong \mathbb{Z}$ with integral coefficients [13]. Now, we are ready to define the simplicial volume of open manifolds.

Definition 2.3. Let M be a connected n-dimensional manifold without boundary. Then, the simplical volume of M is defined as

$$\|M\| = \inf\{\|c\|_1 \mid c \in C_n^{\mathrm{lf}}(M) \text{ is a fundamental cycle of } M\}.$$

The simplicial volume of open manifolds is zero in a large number of cases, including the product of three open manifolds [8], locally symmetric manifolds of $\mathbb{Q}$-rank of at least 3 [15].

Gromov [8] introduced another notion of the simplicial volume, so called Lipschitz simplicial volume which is a geometric variant of the ordinary simplicial volume. Let M be an oriented Riemannian manifold. For a locally finite chain $c = \sum_{\sigma \in A} a_\sigma \sigma$, define $\mathrm{Lip}(c)$ as the supremum of the Lipschitz constants of the singular simplices σ with respect to the standard Euclidean metric on the standard simplex.

Definition 2.4. Let M be an n-dimensional, oriented Riemannian manifold. A locally finite chain c is called a Lipschitz fundamental cycle of M when c represents the fundamental class of M and $\mathrm{Lip}(c) < \infty$. The Lipschitz simplicial volume $\|M\|_{\mathrm{Lip}}$ of M is defined as

$$\|M\|_{\mathrm{Lip}} = \inf\{\|c\|_1 \mid c \in C_n^{\mathrm{lf}}(M) \text{ is a Lipschitz fundamental cycle of } M\}.$$

From the definition of Lipschitz simplicial volume, we have the obvious inequality $\|M\| \leq \|M\|_{\mathrm{Lip}}$ for an oriented, Riemannian manifold M. If M is closed, the fundamental cycles in the locally finite chain complex of M involve only a finite number of simplices, and hence, $\|M\| = \|M\|_{\mathrm{Lip}}$. Löh and Sauer [15] prove the proportionality principle for the Lipschitz simplicial volume under the non-positive curvature condition in the non-compact case.

Theorem 2.5 (Löh and Sauer)**.** *Let M and N be complete, non-positively curved Riemannian manifolds of finite volume. Assume that their universal covers are isometric. Then,*

$$\frac{\|M\|_{Lip}}{vol(M)} = \frac{\|N\|_{Lip}}{vol(N)}.$$

By Theorem 2.5, they can show that the Lipschitz simplicial volume of locally symmetric spaces of finite volume and non-compact type is positive and, moreover, they obtain degree theorems for locally symmetric spaces of non-compact type of finite volume, which is originally due to [5].

Theorem 2.6 (Connell and Farb)**.** *Let M be a locally symmetric n-manifold of non-compact type with finite volume. Assume that M has no local direct factors locally isometric to $\mathbb{R}$, $\mathbb{H}^2$ or $SL_3(\mathbb{R})/SO_3(\mathbb{R})$. Then for any complete Riemannian manifold N with finite volume and any proper Lipschitz map $f : N \to M$,*

$$deg(f) \leq C\frac{vol(N)}{vol(M)}$$

where $C > 0$ depends only on n and the smallest Ricci curvatures of N and M.

Note that the results of Connell and Farb, Löh and Sauer hold with the Lipschitz condition on f. If the positivity of the ordinary simplicial volume of M is verified, one can obtain the degree theorem without the Lipschitz condition on f.

3. $\mathbb{Q}$-rank one locally symmetric spaces

In this section, we recall the definitions of arithmetic lattices, $\mathbb{Q}$-rank, and cusp decomposition in $\mathbb{Q}$-rank one locally symmetric spaces. The cusp decomposition in quotient manifolds by arithmetic lattices is crucial to show the presence of the geodesic straightening map on the locally finite chain complex.

Let X be a connected symmetric space of non-compact type. Let G be the the identity component of the isometry group of X. Then, G is a connected semi-simple Lie group with trivial center and no compact factor [7]. We first recall the definition of arithmetic lattices in [18].

Definition 3.1. Let G be a connected semi-simple Lie group with trivial center and no compact factors. Let $\Gamma \subset G$ be a lattice. Then, Γ is called *arithmetic* if there exist

(i) a semi-simple algebraic group $\mathbf{G} \subset GL(n, \mathbb{C})$ defined over $\mathbb{Q}$ and
(ii) a surjective homomorphism $\rho : \mathbf{G}(\mathbb{R})^0 \to G$ with compact kernel

such that $\rho(\mathbf{G}(\mathbb{Z}) \cap \mathbf{G}(\mathbb{R})^0)$ and Γ are commensurable.

The $\mathbb{Q}$-rank(Γ) is defined as the dimension of any maximal $\mathbb{Q}$-split torus of $\mathbf{G}(\mathbb{Q})$ when Γ is an arithmetic lattice. The structure of the ends of $M = \Gamma\backslash X$ is closely related to the $\mathbb{Q}$-rank(Γ). For instance, a locally symmetric space $M = \Gamma\backslash X$ is compact if and only if the $\mathbb{Q}$-rank(Γ) is zero by the result of Borel and Harish-Chandra. To understand the ends of quotient manifolds by arithmetic lattices, we recall the reduction theory due to A. Borel and Harish-Chandra [1].

Theorem 3.2 (Borel, Harish-Chandra)**.** *Let $\mathbf{G}$ be a semi-simple algebraic group defined over $\mathbb{Q}$ with associated Riemannian symmetric space X. Let $\mathbf{P}$ be a minimal parabolic $\mathbb{Q}$-subgroup of $\mathbf{G}$ and let Γ be an arithmetic subgroup of $\mathbf{G}(\mathbb{Q})$. Then*

(i) *the set of double cosets $\mathcal{F} = \Gamma\backslash\mathbf{G}(\mathbb{Q})/\mathbf{P}(\mathbb{Q})$ is finite,*
(ii) *there exists a generalized Siegel set $\mathcal{S}_{\omega,\tau}$ such that for a (fixed) set $\{q_i \mid 1 \le i \le m\}$ of representatives of $\mathcal{F}$ the union $\Omega = \bigcup_{i=1}^m q_i \cdot \mathcal{S}_{\omega,\tau}$ is a fundamental set for Γ in X.*

By using the reduction theory, Leuzinger [11] gives an explicit differential geometric description of M.

Theorem 3.3 (Leuzinger)**.** *Let X be a Riemannian symmetric space of non-compact type and with $\mathbb{R}$-rank ≥ 2 and let Γ be an irreducible, torsion-free, non-uniform lattice in the isometry group of X. On the locally symmetric space $M = \Gamma\backslash X$ there exists a continuous and piecewise real analytic exhaustion function $h : M \to [0, \infty)$ such that, for any $s \ge 0$, the sublevel set $M(s) = \{h \le s\}$ is a compact submanifold with corners of M. Moreover the boundary of $M(s)$, which is a level set of h, consists of projections of subsets of horospheres in X.*

More precisely, there exists a union of a countable number of open horoballs $U(s)$ of X such that $X(s) = X - U(s)$ is Γ-invariant and $M(s) = \Gamma\backslash X(s)$. If $s_1 < s_2$, then $U(s_2) \subset U(s_1)$ and the subsets $X(s) = X - U(s)$ exhaust X. These horoballs are in one-to-one correspondence with the vertices of the

Tits building of $\mathbf{G}$ over $\mathbb{Q}$. The deleted horoballs are disjoint if and only if Γ is an arithmetic subgroup of a semisimple algebraic group of $\mathbb{Q}$-rank one. In the case of higher $\mathbb{Q}$-rank, the horoballs of $U(s)$ intersect and give rise to corners [12].

Indeed, the theorem of Leuzinger is available for torsion-free arithmetic lattices with $\mathbb{Q}$-rank at least one because the proof in the paper [11] is only based on the reduction theory for arithmetic lattices. He used the arithmeticity theorem of Margulis to show that all irreducible, torsion-free, non-uniform lattices with $\mathbb{R}$-rank≥ 2 are arithmetic lattices.

4. Geodesic straightening map on locally finite chain complex

The geodesic straightening map has played an important role in proving the positivity of the simplicial volume. In this section, we will show the presence of the geodesic straightening map on the locally finite chain complex of $\mathbb{Q}$-rank one locally symmetric spaces.

4.1. Geodesic Straightening.

The geodesic straightening map on the level of singular chain complexes was introduced by Thurston [17]. We recall the definition of the geodesic straightening map.

Let X be a simply connected, complete Riemannian manifold with non-positive sectional curvature. For $x_0, \ldots, x_k \in X$, the geodesic simplex $[x_0, \ldots, x_k]$ is defined inductively as follows. First, $[x_0]$ is the point $x_0 \in X$, and $[x_0, x_1]$ is the unique geodesic arc from x_1 to x_0. In general, $[x_0, \ldots, x_k]$ is the geodesic cone on $[x_0, \ldots, x_{k-1}]$ with the top point x_k.

Definition 4.1. Let M be a connected, complete Riemannian manifold with non-positive sectional curvature. Then, the geodesic straightening map $st_* : C_*(M) \to C_*(M)$ is defined by

$$st_k(\sigma) = \pi_M \circ [\tilde{\sigma}(e_0), \ldots, \tilde{\sigma}(e_k)] \text{ for a singular } k\text{-simplex } \sigma,$$

where $\pi_M : \widetilde{M} \to M$ is the universal covering, $e_0, \ldots, e_k$ are the vertices of the standard k-simplex Δ^k, and $\widetilde{\sigma}$ is a lift of σ to the universal cover $\widetilde{M}$.

The following proposition proved by Thurston makes it possible to obtain the simplicial volume of M by considering only the ℓ^1-norm on the geodesically straight chains of M.

Proposition 4.2 (Thurston). *Let M be a connected, complete Riemannian manifold with non-positive sectional curvature. Then, the geodesic straightening map is chain homotopic to the identity.*

4.2. Straightening locally finite chains.

Let M be a $\mathbb{Q}$-rank one locally symmetric space. First, we fix some notations. Let X denote the universal cover of M and Γ the fundamental group of M. Let $h : M \to [0, \infty)$ be the exhaustion function of M under Leuzinger's theorem in [11]. For any $s \geq 0$, M admits the following disjoint decomposition

$$M = M(s) \cup \coprod_{i=1}^{l} E_i(s),$$

where the sublevel set $M(s) = \{h \leq x\}$ is a compact submanifold and $E_i(s)$ is a cusp end of $M - M(s)$ for each $i = 1, \ldots, l$. As we already mentioned,

there is a countable number of pairwise disjointed horoballs $U(s)$ in X such that each $E_i(s)$ is obtained by the quotient of an open horoball in $U(s)$ by Γ and $M(s) = \Gamma\backslash(X - U(s))$. Thus, $E_i(s)$ is geodesically convex for each $i = 1, \ldots, l$.

Lemma 4.3. *Let M be a $\mathbb{Q}$-rank one locally symmetric space. Then, the geodesic straightening of a locally finite chain in $C_*^{lf}(M)$ is a locally finite chain.*

Proof. Let $A \in S_k^{\mathrm{lf}}(M)$ and $c = \sum_{\sigma \in A} a_\sigma \sigma$ be a locally finite chain in $C_k^{\mathrm{lf}}(M)$. Let us define $st_k(A) = \{st_k(\sigma) \mid \sigma \in A\}$. It is sufficient to prove that $st_k(A)$ is locally finite. Let K be a compact subset of M. Then, one can choose a compact submanifold $M(s)$ of M containing K for some $s > 0$.

By the local finiteness of A, a compact submanifold $M(s)$ intersects the image of only a finite number of elements of A. Let σ be an element of A whose image does not intersect $M(s)$. Then, we claim that the image of σ has to be contained in only one cusp end of $M - M(s)$. Suppose the image of σ intersects at least two cusp ends of $M - M(s)$, denoted by $E_1(s)$ and $E_2(s)$. Since the image of σ is path-connected, there is a path in the image of σ connecting two different points contained in $E_1(s)$ and $E_2(s)$, respectively. However, any path connecting such two points must pass through $M(s)$. This means that the image of σ intersects $M(s)$, which contradicts the assumption that the image of σ does not intersect $M(s)$.

Now, let's assume that the image of σ is contained in $E_1(s)$. Since $E_1(s)$ is geodesically convex and the image of σ is contained in $E_1(s)$, the image of geodesic straightening $st_k(\sigma)$ of σ is also totally contained in $E_1(s)$. This implies that the image of $st_k(\sigma)$ does not intersect both $M(s)$ and K. Hence, we conclude that K can intersect the image of $st_k(\tau)$ for only a finite number of elements τ of A intersecting $M(s)$, and so $st_k(A)$ is locally finite. This completes the proof of the lemma. □

By Lemma 4.3, the geodesic straightening map is well-defined on the locally finite chain complex of M:

$$st_*^{\mathrm{lf}} : C_*^{\mathrm{lf}}(M) \to C_*^{\mathrm{lf}}(M).$$

The map st_*^{lf} is obviously a chain map because it is induced from the geodesic straightening map on the singular chain complex of M. Furthermore, we prove that it is chain homotopic to the identity as follows.

Proposition 4.4. *Let M be a $\mathbb{Q}$-rank one locally symmetric space. Then the geodesic straightening map st_*^{lf} is chain homotopic to the identity.*

Proof. First, we recall the construction of the chain homotopy $H_* : C_*(M) \to C_{*+1}(M)$ from the geodesic straightening map to the identity. The chain homotopy H_k is defined by the straight line homotopy between any k-simplex and its geodesically straight simplex. Moreover, these homotopies, when restricted to lower dimentional faces, agree with the homotopies canonically defined on those faces. For more details, let H_σ be a canonical straight line homotopy

$$H_\sigma : \Delta^k \times [0,1] \to M$$

from σ to $st_k(\sigma)$ for any k-simplex σ. Now $\Delta^k \times [0,1]$ has vertices

$$a_0 = (e_0, 0), \dots, a_k = (e_k, 0), b_0 = (e_0, 1), \dots, b_k = (e_k, 1).$$

For each $i = 0, \dots, k$, let

$$\alpha_i : \Delta^{k+1} \to \Delta^k \times [0,1]$$

be the affine map that maps $e_0, \dots, e_{k+1}$ to $a_0, \dots, a_i, b_i, \dots, b_k$, respectively. Define linear transformation

$$H_k : C_k(M; \mathbb{R}) \to C_{k+1}(M; \mathbb{R})$$

by the formula

$$H_k(\sigma) = \sum_{i=0}^{k} (-1)^i H_\sigma \circ \alpha_i.$$

This H_* is the chain homotopy from the geodesic straightening map to the identity on the singular chain complex of M.

Let $c = \sum_{\sigma \in A} a_\sigma \sigma$ be a locally finite chain. We claim that $H_*(c)$ is a locally finite chain again. Let K be a compact subset of M. Choose a compact submanifold $M(s)$ containing K for some $s > 0$. Suppose that the image of $\sigma \in A$ does not intersect $M(s)$. By a similar argument in Lemma 4.3, the images of both σ and $st_k(\sigma)$ are totally contained in only one cusp end, denoted by $E(s)$.

Since $E(s)$ is geodesically convex, the image of the straight line homotopy $H_k(\sigma)$ between σ and $st_k(\sigma)$ is totally contained in $E(s)$. This means that $M(s)$ does not intersect the image of $H_k(\sigma)$ and K does not either. Therefore, K can intersect the image of $H_k(\sigma)$ for only a finite number of elements σ of A intersecting $M(s)$. Since $H_k(\sigma)$ is a finite sum of simplices, K intersects the image of only a finite number of simplices in $H_k(c)$. In other words, $H_k(c)$ is a locally finite chain. Finally, we obtain the following well-defined map:

$$H_k^{\mathrm{lf}} : C_k^{\mathrm{lf}}(M) \to C_{k+1}^{\mathrm{lf}}(M)$$

satisfying

$$\partial_{k+1} H_k^{\mathrm{lf}} + H_{k-1}^{\mathrm{lf}} \partial_k = st_k^{\mathrm{lf}} - id.$$

Therefore, H_*^{lf} is a chain homotopy from st_*^{lf} to the identity. □

As can be seen in the proof of Lemma 4.3 and Proposition 4.4, $\mathbb{Q}$-rank one condition on M is essential to obtain the geodesic straightening map on the locally finite chain complex of M. Since the cusp end of higher $\mathbb{Q}$-rank locally symmetric space has corners, it is not geodesically convex. Hence, Lemma 4.3 and Proposition 4.4 fail in the case of a higher $\mathbb{Q}$-rank locally symmetric space.

5. Positivity of the simplicial volume

Now, we will prove the positivity of the simplicial volume of $\mathbb{Q}$-rank one locally symmetric spaces M covered by the product of $\mathbb{R}$-rank one symmetric spaces.

Let X_i be a complete, simply-connected, n_i-dimensional, Riemannian manifold with negative sectional curvature bounded away from zero for each $i = 1, \dots, k$. Let X be the n-dimensional product manifold $X_1 \times \cdots \times X_k$.

Now, we prove that the volume of geodesic n-simplices in X is uniformly bounded from above.

Lemma 5.1. *The volume of geodesic n-simplices in X is uniformly bounded from above.*

Proof. Let $[x_0, \dots, x_n]$ be a geodesic n-simplex for an ordered vertex set $\{x_0, \dots, x_n\} \subset X$. Let $p_i : X_1 \times \cdots \times X_k \to X_i$ denote the projection map from X onto X_i for each $i = 1, \dots, k$. Then, we have

$$\mathrm{Vol}([x_0, \dots, x_n]) \leq \prod_{i=1}^{k} \mathrm{Vol}(p_i[x_0, \dots, x_n]). \tag{1}$$

From Equation (1), it suffices to show that the volume of $p_i[x_0, \dots, x_n]$ in X_i is uniformly bounded from above for each $i = 1, \dots, k$. First, note that $p_i([x_0, \dots, x_n]) = [p_i(x_0), \dots, p_i(x_n)]$ because a geodesic in X projects to a geodesic in X_i by the projection map $p_i : X \to X_i$. In other words, $p_i([x_0, \dots, x_n])$ is a geodesic n-simplex in X_i. When $n \geq n_i$, a geodesic n-simplex in X_i consists of at most $\binom{n}{n_i}$ geodesic n_i-simplices in X_i. More precisely, for a geodesic n-simplex $[y_0, \dots, y_n]$ in X_i, we have

$$[y_0, \dots, y_n] = \bigcup_{0 \leq l_0 < \cdots < l_{n_i} \leq n} [y_{l_0}, \dots, y_{l_{n_i}}].$$

Since the volume of geodesic n_i-simplices in X_i is uniformly bounded from above [9], so is the volume of geodesic n-simplices in X_i. Each uniform upper bound on the volumes of geodesic n-simplices in X_i for each $i = 1, \dots, k$ gives a uniform upper bound on the volumes of geodesic n-simplices in X by Equation (1), which completes the proof. $\square$

We now prove the main theorems. Let M be an n-dimensional Riemannian manifold. Then, the evaluation map

$$\langle \cdot, \cdot \rangle : C^*(M) \otimes C_*(M) \longrightarrow \mathbb{R}$$

is well-defined and it induces the Kronecker product on $H^*(M) \otimes H_*(M)$. Let $K \subset M$ be a compact, connected subset with non-empty interior. Let $\Omega^*(M, M-K)$ be the kernel of the restriction homomorphism $\Omega^*(M) \to \Omega^*(M-K)$ on differential forms. The corresponding cohomology groups are denoted by $H^*_{dR}(M, M-K)$. The de Rham map $\Omega^*(M) \to C^*(M)$ restricts to the respective kernels and, thus, induces a homomorphism, called relative de Rham map,

$$\Psi^* : H^*_{dR}(M, M-K) \to H^*(M, M-K).$$

The relative de Rham map is an isomorphism. Note that integration gives a homomorphism $\int : H^*_{dR}(M, M-K) \to \mathbb{R}$. Moreover, it is well-known that

$$\langle \Psi^n[\omega], [M, M-K] \rangle = \int_M \omega$$

holds for all n-forms ω [6].

Proposition 5.2. *Let M be a n-dimensional, $\mathbb{Q}$-rank one locally symmetric space covered by the product of $\mathbb{R}$-rank one symmetric spaces. Let $c = \sum_{k\in\mathbb{N}} a_k\sigma_k$ be a fundamental cycle of M with $\|c\|_1 < \infty$. Then,*

$$\sum_{k\in\mathbb{N}} a_k \cdot \langle dvol_M, st_n(\sigma_k)\rangle = vol(M).$$

Proof. Geodesic simplices in M are C^1, and the volume of geodesic simplices in M is uniformly bounded from above by Lemma 5.1. Hence, there exists a uniform constant $C > 0$ such that we have

$$|\langle dvol_M, st_n(\sigma)\rangle| \leq C,$$

for any singular simplex σ in M. From this inequality, one can see that $\sum_{k\in\mathbb{N}} a_k \cdot \langle dvol_M, st_n(\sigma_k)\rangle$ converges absolutely. Since st_*^{lf} is chain homotopic to the identity, $[st_n^{\mathrm{lf}}(c)]$ is also a fundamental class of M.

Let $K \subset M$ be a connected, compact subset with non-empty interior. For $\delta \in \mathbb{R}_{>0}$, let $g_\delta : M \to [0,1]$ be a smooth function supported on the closed δ-neighborhood K_δ of K with $g_\delta|_K = 1$. Then, $g_\delta dvol_M \in \Omega^n(M, M - K_\delta)$ is a cocyle and

$$vol(K) = \lim_{\delta\to 0} \int_M g_\delta dvol_M.$$

The map $H_n(i_\delta) : H_n^{\mathrm{lf}}(M) \to H_n(M, M - K_\delta)$ induced by the inclusion $i_\delta : (M, \phi) \to (M, M - K_\delta)$ maps the fundamental class of M to the relative fundamental class $[M, M - K_\delta]$ of $(M, M - K_\delta)$, and $H_n(i_\delta)[st_n^{\mathrm{lf}}(c)]$ is represented by $\sum_{im\sigma_k \cap K_\delta \neq \phi} a_k st_n(\sigma_k)$. Since $H_n(i_\delta)[st_n^{\mathrm{lf}}(c)]$ is also the relative fundamental class of $(M, M - K_\delta)$, we have

$$\begin{aligned}
&\lim_{\delta\to 0} \sum_{im\sigma_k\cap K_\delta\neq\phi} a_k \cdot \langle g_\delta dvol_M, st_n(\sigma_k)\rangle \\
&= \lim_{\delta\to 0} \langle \Psi^n[g_\delta dvol_M], [M, M - K_\delta]\rangle \\
&= \lim_{\delta\to 0} \int_M g_\delta dvol_M \\
&= vol(K).
\end{aligned}$$

For each $k \in \mathbb{N}$ and $\delta \in \mathbb{R}_{>0}$, we also have a uniform upper bound

$$|\langle g_\delta dvol_M, st_n(\sigma_k)\rangle| \leq C,$$

and hence,

$$\Big| \sum_{k\in\mathbb{N}} a_k \cdot \langle dvol_M, st_n(\sigma_k)\rangle - \sum_{im\sigma_k\cap K_\delta\neq\phi} a_k \cdot \langle g_\delta dvol_M, st_n(\sigma_k)\rangle \Big| \leq 2C \sum_{im\sigma_k \subset M-K} |a_k|.$$

Because $\sum_{k\in\mathbb{N}} |a_k| < \infty$, there is an exhausting sequence $(K^m)_{m\in\mathbb{N}}$ of compact, connected subsets of M with non-empty interior satisfying

$$\lim_{m\to\infty} vol(K^m) = vol(M) \text{ and } \lim_{m\to\infty} \sum_{im\sigma_k\subset M-K^m} |a_k| = 0.$$

Thus, the estimates of the previous paragraphs yield

$$\begin{aligned}\sum_{k\in\mathbb{N}} a_k \cdot \langle dvol_M, st_n(\sigma_k)\rangle &= \lim_{m\to\infty}\lim_{\delta\to\infty} \sum_{im\sigma_k\cap K^m_\delta\neq\phi} a_k\cdot\langle g^m_\delta dvol_M, st_n(\sigma_k)\rangle \\ &= \lim_{m\to\infty} vol(K^m) \\ &= vol(M),\end{aligned}$$

which establishes the formula. □

From Proposition 5.2, the positivity of the simplicial volume of $\mathbb{Q}$-rank one locally symmetric spaces covered by the product of $\mathbb{R}$-rank one symmetric spaces is directly obtained as follows.

Theorem 5.3. *Let M be a $\mathbb{Q}$-rank one locally symmetric space covered by the product of $\mathbb{R}$-rank one symmetric spaces. Then, the simplicial volume of M is positive.*

Proof. Let $c=\sum_{k\in\mathbb{N}} a_k\sigma_k \in C^{\mathrm{lf}}_n(M)$ be a fundamental cycle. From Proposition 5.2, we have

$$vol(M)=\sum_{k\in\mathbb{N}} a_k\cdot\langle dvol_M, st_n(\sigma_k)\rangle.$$

A uniform upper bound of the volume of geodesic simplices in M yields the inequality

$$vol(M)\leq C\cdot\sum|a_k|,$$

where $C>0$ depends only on the universal cover of M. Dividing and passing to the infimum over all fundamental cycles, it provides the positive lower bound

$$\|M\|\geq vol(M)/C>0.$$

Therefore, we conclude that the simplicial volume of M is positive. □

Gromov [8] provided a lower bound for the minimal volume minvol(M), which is defined as the infimum of volumes over all complete Riemannian metrics on M with sectional curvatures bounded between -1 and 1 in terms of the simplicial volume of an n-dimensional smooth manifolds M:

$$\|M\|\leq (n-1)^n n!\cdot \mathrm{minvol}(M). \tag{2}$$

By Inequality (2) and Theorem 5.3, we have the following corollary.

Corollary 5.4. *The minimal volume of $\mathbb{Q}$-rank one locally symmetric spaces covered by the product of $\mathbb{R}$-rank one symmetric spaces is positive.*

See [4, 5] for different approach using Lipschitz maps.

6. Degree theorem

For any proper map

$$f:N\to M$$

between finite volume Riemannian manifolds, a locally finite fundamental cycle is mapped to a locally finite cycle. Hence, usual inequality

$$deg(f)\cdot||M||\leq||N|| \tag{3}$$

holds.

Theorem 6.1. *Let N be a Riemannian n-dimensional manifold of finite volume with Ricci curvature bounded below by $-(n-1)$ and M be a $\mathbb{Q}$-rank one locally symmetric space covered by the product of $\mathbb{R}$-rank one symmetric spaces. For any proper map $f : N \to M$ we have*

$$deg(f) \leq C_n \frac{vol(N)}{vol(M)},$$

where C_n depends only on n.

Proof. By Gromov [8],

$$||N|| \leq (n-1)^n n! vol(N).$$

Since

$$vol(M) \leq C \cdot ||M||,$$

from Equation (3), we get

$$\frac{deg(f)}{C} vol(M) \leq deg(f)||M|| \leq ||N|| \leq (n-1)^n n! vol(N).$$

Since, for a given dimension n, there are only finitely many symmetric spaces, the constant C depends only on n. $\square$

Such a degree theorem is known by [5, 15] for proper Lipschitz map f with the sectional curvature of N bounded above by 1 and any n-dimensional locally symmetric manifold M of finite volume. Note that they obtained the degree theorem for proper Lipschitz map f by verifying the positivity of the Lipschitz simplicial volume of M. Our result about the positivity of the ordinary simplicial volume of M yields the degree theorem without any Lipschitz condition on map f.

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